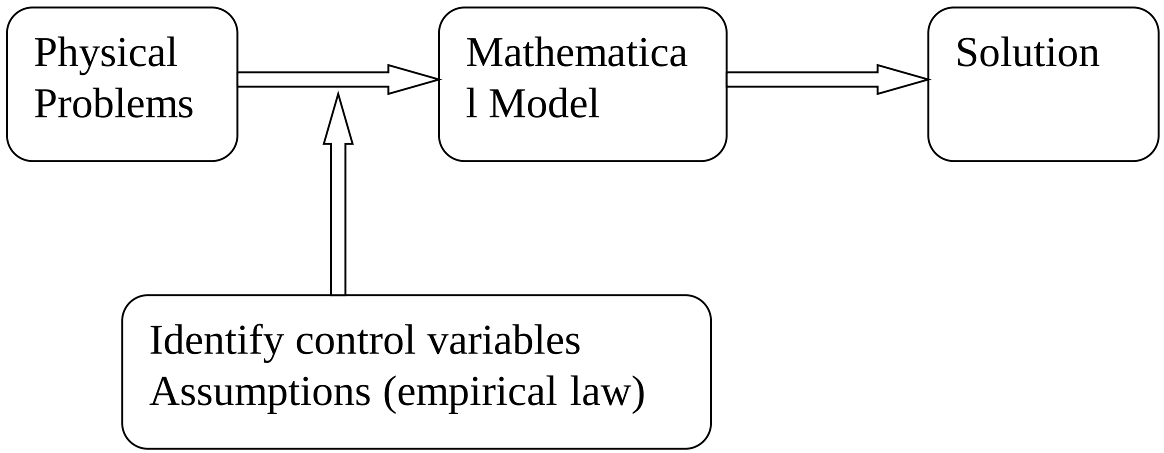
Introduction to Finite Element Methods

Introduction to Finite Element Method

1. Mathematical Model

(1) Modeling



(2) Types of solution

Sol. \ Eq.	Exact Eq.	Approx. Eq.
Exact Sol.	◎	◎
Approx. Sol.	◎	◎

(3) Methods of Solution

- Engineering Analysis
  - Classical method
    - Exact Solution
    - Approximate Solution
  - Numerical method
    - Energy
    - Boundary element
    - Finite Difference
    - Finite element 有限元素

(3) Method of Solution

A. Classical methods

They offer a high degree of insight, but the problems are difficult or impossible to solve for anything but simple geometries and loadings.

B. Numerical methods

(I) Energy: Minimize an expression for the potential energy of the structure over the whole domain.

(II) Boundary element: Approximates functions satisfying the governing differential equations not the boundary conditions.

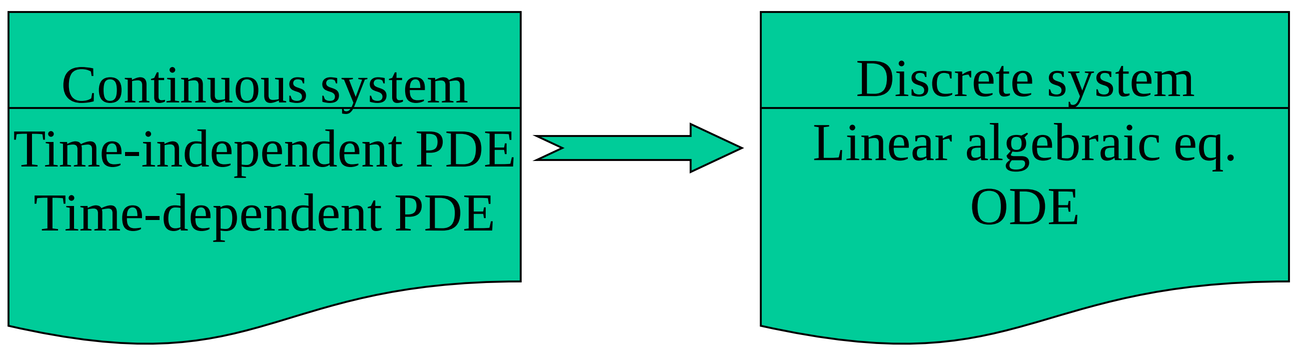
(III) Finite difference: Replaces governing differential equations and boundary conditions with algebraic finite difference equations.

(IV) Finite element: Approximates the behavior of an irregular, continuous structure under general loadings and constraints with an assembly of discrete elements.

2. Finite Element Method

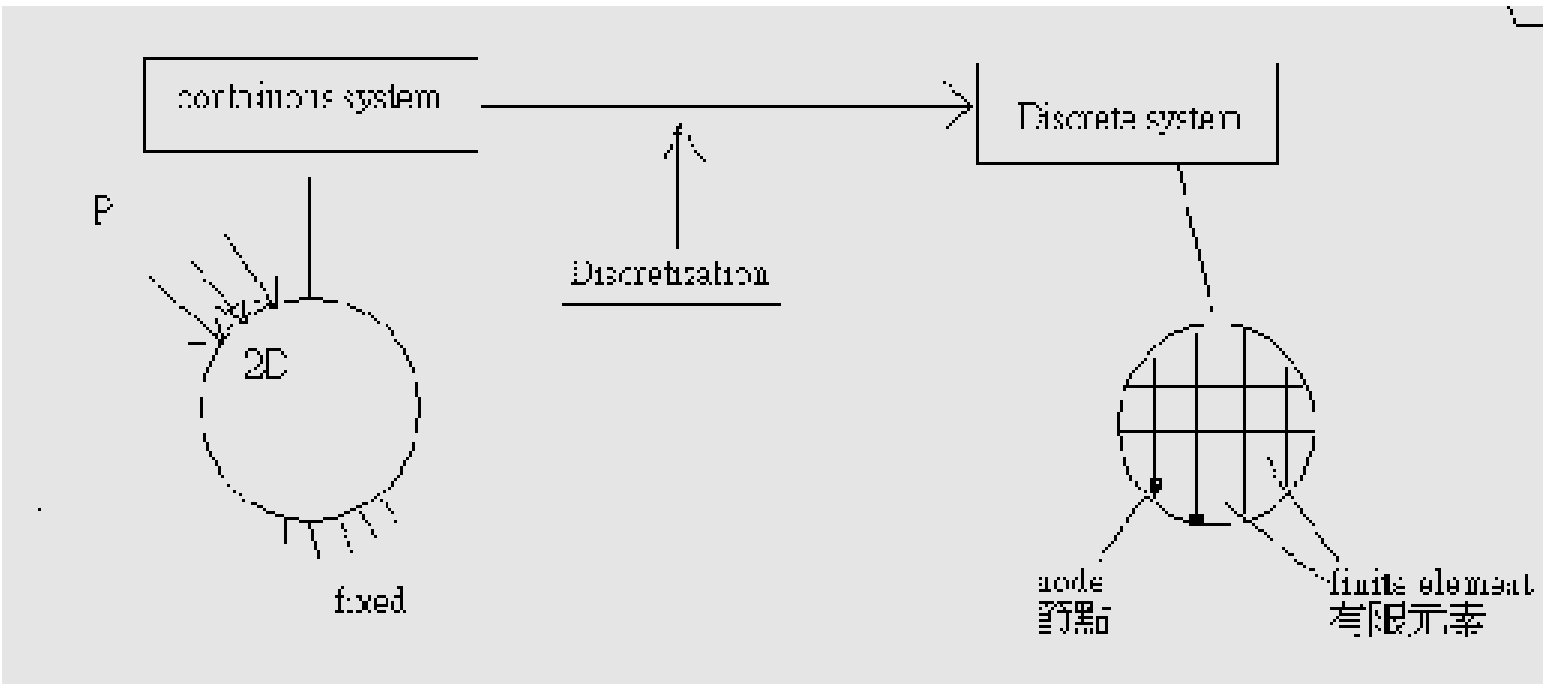
(1) Definition

FEM is a numerical method for solving a system of governing equations over the domain of a continuous physical system, which is discretized into simple geometric shapes called finite element.



(2) Discretization

Modeling a body by dividing it into an equivalent system of finite elements interconnected at a finite number of points on each element called nodes.



3. Historical Background

1940 →
- Structural engineering
- One dimensional element

1950 →
- Direct stiffness method
- two dimensional elements

1960 →
- "Finite element" introduced

1970 →
- three dimensional elements
- Large deflection
- Thermal analysis

1976 — Large deformation nonlinear dynamics

Present — New elements, New softwares.

Chronicle of Finite Element Method

Year	Scholar	Theory
1941	Hrennikoff	Presented a solution of elasticity problem using one-dimensional elements.
1943	McHenry	Same as above.
1943	Courant	Introduced shape functions over triangular subregions to model the whole region.
1947	Levy	Developed the force (flexibility) method for structure problem.
1953	Levy	Developed the displacement (stiffness) method for structure problem.
1954	Argyris & Kelsey	Developed matrix structural analysis methods using energy principles.
1956	Turner, Clough, Martin, Topp	Derived stiffness matrices for truss, beam and 2D plane stress elements. Direct stiffness method.
1960	Clough	Introduced the phrase finite element .
1960	Turner et. al	Large deflection and thermal analysis.
1961	Melosh	Developed plate bending element stiffness matrix.
1961	Martin	Developed the tetrahedral stiffness matrix for 3D problems.
1962	Gallagher et al	Material nonlinearity.

Chronicle of Finite Element Method

Year	Scholar	Theory
1963	Grafton, Strome	Developed curved-shell bending element stiffness matrix.
1963	Melosh	Applied variational formulation to solve nonstructural problems.
1965	Clough et. al	3D elements of axisymmetric solids.
1967	Zienkiewicz et.	Published the first book on finite element.
1968	Zienkiewicz et.	Visco-elasticity problems.
1969	Szabo & Lee	Adapted weighted residual methods in structural analysis.
1972	Oden	Book on nonlinear continua.
1976	Belytschko	Large-displacement nonlinear dynamic behavior.
~1997		New element development, convergence studies, the developments of supercomputers, the availability of powerful microcomputers, the development of user-friendly general-purpose finite element software packages.

4. Analytical Processes of Finite Element Method

(1) Structural stress analysis problem

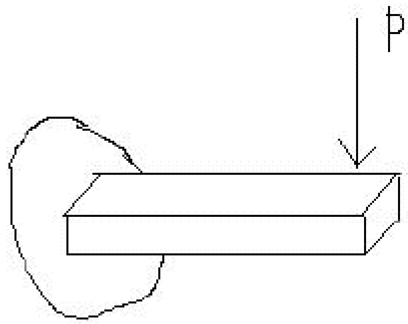
A. Conditions that solution must satisfy
   a. Equilibrium
   b. Compatibility
   c. Constitutive law
   d. Boundary conditions

Above conditions are used to generate a system of equations representing system behavior.

B. Approach
   a. Force (flexibility) method: internal forces as unknowns.
   b. Displacement (stiffness) method: nodal disp. As unknowns.

For computational purpose, the displacement method is more desirable because its formulation is simple. A vast majority of general purpose FE softwares have incorporated the displacement method for solving structural problems.

(2) Analysis procedures of linear static structural analysis



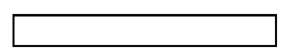
1D problem ?
2D problem ?
3D problem ?

A. Build up geometric model

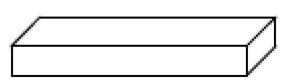
a. 1D problem

line

b. 2D problem

surface 

c. 3D problem

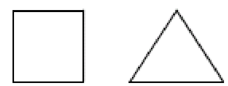
solid 

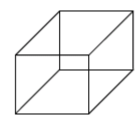
B. Construct the finite element model

a. Discretize and select the element types

(a) element type

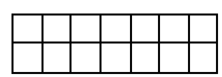
1D line element

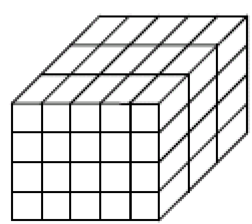
2D element 

3D brick element 

(b) total number of element (mesh)

1D:

2D: 

3D: 

b. Select a shape function

1D line element: u=ax+b

c. Define the compatibility and constitutive law

$$1\text{D}：\ \varepsilon x = \frac{du}{dx} \qquad \sigma = E\varepsilon \quad \text{虎克定律}$$

d. Form the element stiffness matrix and equations

(a) Direct equilibrium method

(b) Work or energy method

(c) Method of weight Residuals

$$[K]^e\{d\}^e = \{F\}^e$$

e. Form the system equation

Assemble the element equations to obtain global system equation and introduce boundary conditions

$$[K]\{d\} = \{F\}$$

C. Solve the system equations

a. elimination method

Gauss’s method (Nastran)

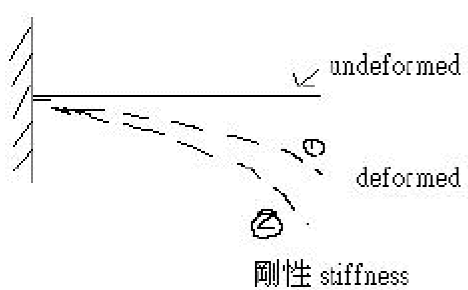
b. iteration method

Gauss Seidel’s method

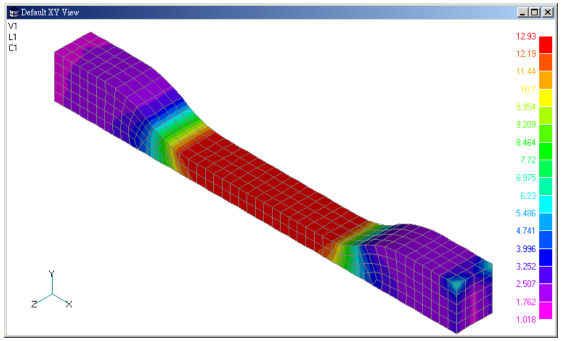
Displacement field → strain field → stress field

D. Interpret the results (postprocessing)

a. deformation plot



b. stress contour



5. Applications of Finite Element Method

Structural Problem	Non-structural Problem
Stress Analysis - truss & frame analysis - stress concentrated problem Buckling problem Vibration Analysis Impact Problem	Heat Transfer Fluid Mechanics Electric or Magnetic Potential

6. Computer Programs for Finite Element Method

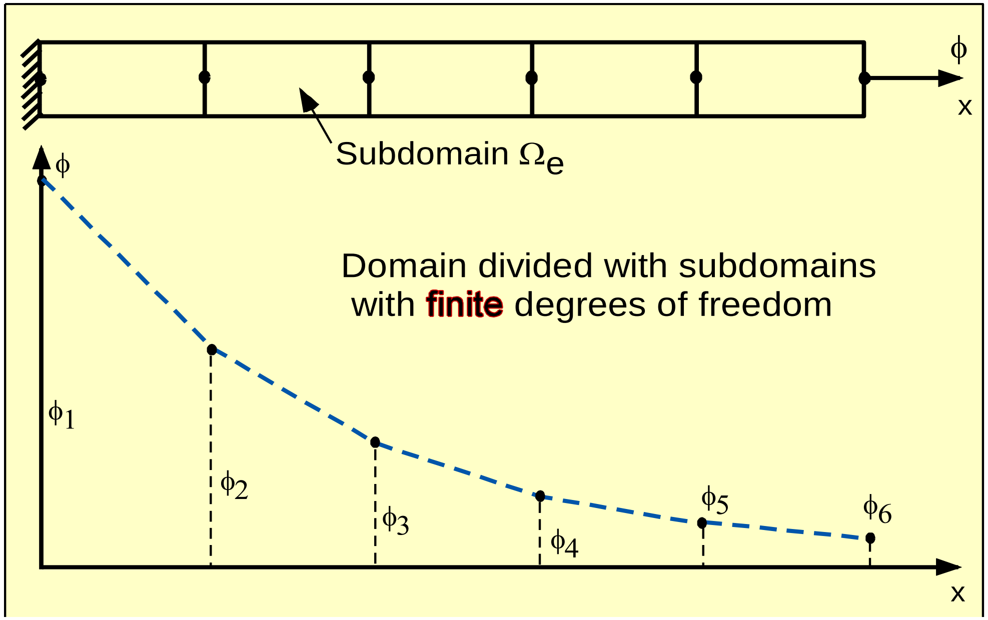
ANSYS	◎	Δ	◎	◎		Δ		◎	◎		◎
NASTRAN	◎	Δ	◎	◎		Δ		◎			◎
ABAQUS	◎	◎			◎			◎			◎
MARC	◎	◎			◎			◎			◎
LS-DYNA3D					◎						
MSC/DYNA					◎						
ADAMS/ DADS							◎				
COSMOS	◎	Δ	◎	◎		Δ			◎		◎
MOLDFLOW										◎	
C-FLOW										◎	
PHOENICS									◎		◎

Finite Element Method (FEM)

- A continuous function of a continuum (given domain Ω) having infinite degrees of freedom is replaced by a discrete model, approximated by a set of piecewise continuous functions having a finite degree of freedom.

General Example

- A bar subjected to some excitations like applied force at one end. Let the field quantity flow through the body, which has been obtained by solving governing DE/PDE, In FEM the domain Ω is subdivided into sub domain and in each sub domain a piecewise continuous function is assumed.



General Steps of the FEM

- 1. Discretize & Select the Element Types
- 2. Select a Displacement Function
- 3. Define the Strain/Displacement & Stress/Strain
- Relationships
- 4. Derive the Element Stiffness Matrix & Equations
- 5. Assemble the Element Equations to Obtain the Global
- & Introduce Boundary Conditions
- 6. Solve for the Unknown Degrees of Freedom
- 7. Solve for the Element Strains & Stresses
- 8. Interpret the Results

Discretize & Select the Element Types

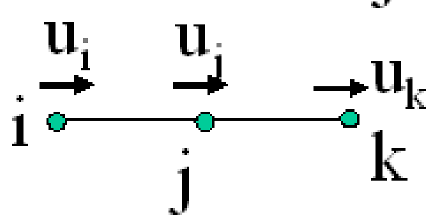
- Divide the body into equivalent systems of finite elements with nodes and the appropriate element type
- Element Types:
  - One-dimensional (Line) Element
  - Two-dimensional Element
  - Three-dimensional Element
  - Axisymmetric Element

One Dimensional Element

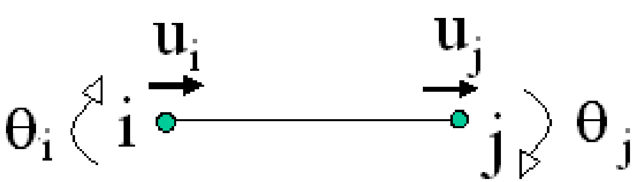
Select a Displacement Function

- There will be a displacement function for each element

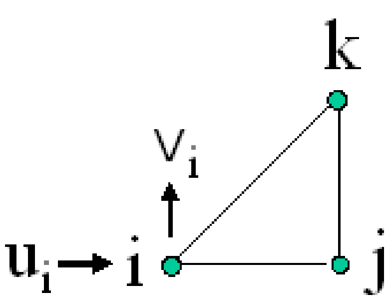
$$u(x) = \alpha_1 + \alpha_2 x$$



$$u(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2$$



$$U(x) = \alpha_1 + \alpha_2 \cdot x + \alpha_3 \cdot x^2 + \alpha_4 \cdot x^3$$



$$U(x, y) = \alpha_1 + \alpha_2 \cdot x + \alpha_3 \cdot y$$

$$V(x, y) = \alpha_4 + \alpha_5 \cdot x + \alpha_6 \cdot y$$

Pascal's Triangle

1	Constant
$x \quad y$	Linear
$x^2 \quad xy \quad y^2$	Quadratic
$x^3 \quad x^2y \quad xy^2 \quad y^3$	Cubic
$x^4 \quad x^3y \quad x^2y^2 \quad xy^3 \quad y^4$	Quartic

Define Strain Displacement & Stress/Strain Relationships

- For one-dimensional;
  Deformation in the x-direction, strain ε is related to the displacement υ

$$\varepsilon_x = \frac{d}{dx} u$$

$$\varepsilon = \frac{du}{dx} = \frac{d[N]}{dx}\{d\} = [B]\{d\}$$

- [B] – Matrix relating strain to nodal displacement

$$[B] = \frac{1}{L}\begin{bmatrix} -1 & 1 \end{bmatrix}$$

- Hooke's Law is used for the stress/strain relationship

$$\sigma_x = E \cdot \varepsilon_x$$

$\sigma_x = E\varepsilon_x$ To Stiffness Matrix

$$\sigma_x = E \cdot \varepsilon_x$$

$$\sigma_x = \frac{P}{A} \qquad E = \text{Youngs Modulus} \qquad \varepsilon_x = \frac{d}{dx}u$$

$$P = A \cdot E \cdot \frac{d}{dx}u$$

When viewing from u1 to u2

$$\frac{d}{dx}u = u_1 - u_2$$

When viewing from u2 to u1

$$\frac{d}{dx}u = u_2 - u_1$$

When combining the two together for the one element you obtain the stiffness matrix

$$\mathbf{k}_1 = \frac{EA}{L} \begin{array}{c} \begin{array}{cc} u_1 & u_2 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \end{array}$$

Derive the Element Stiffness Matrix & Equations

- Virtual work principle of a deformable body in equilibrium is subjected to arbitrary virtual displacement satisfying compatibility condition (admissible displacement), then the virtual work done by external loads will be equal to virtual strain energy of internal stresses.
- δU^e is the element internal energy
- δW^e is the element external energy
- Please view the integration sheet

$$\delta U^e = \delta W^e$$

Stiffness Matrix

- f_e – Element Force
- k_e – Element Stiffness Matrix
- d^e – Element Displacement

$$\{f_e\} = [K_e]\{d^e\}$$

- E – Young Modulus
- A – Cross Section Area
- L - Length

$$\mathbf{k}_1 = \frac{EA}{L}\begin{array}{c c} & \begin{array}{cc} u_1 & u_2 \end{array} \\ & \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{array}{c} 1 \\ 2 \end{array} \end{array}$$

Assemble Equations for Global Matrix & Introduce Boundary Conditions

- Combine each element stiffness matrix into one, which is known as the global matrix
- This is done by combining each $[k_e]$ into their proper location on the global matrix

$$[K_e] = \int_{v^e} [B]^T [D][B] dv$$

$$\{F\} = [K]\{D\}$$

- Capital letters represent the same as the element stiffness matrix, but for global matrix

Solve for Unknown DOF's

- Using the global matrix with the boundary conditions, we can now eliminate some variables and solve for the unknowns, i.e. displacements, end forces

$$\begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ \ldots\ldots \\ F_n \end{pmatrix} = \begin{pmatrix} k_{11} & k_{12} & k_{13} & \ldots\ldots & k_{1n} \\ k_{21} & k_{22} & k_{23} & \ldots\ldots & k_{2n} \\ k_{31} & k_{32} & k_{33} & \ldots\ldots & k_{3n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ k_{n1} & \ldots\ldots & \ldots\ldots & \ldots\ldots & k_{nn} \end{pmatrix} \cdot \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ \ldots\ldots \\ d_n \end{pmatrix}$$

Solve for Element Strains & Stresses Interpret Results

- Solve for the stress using the equation below

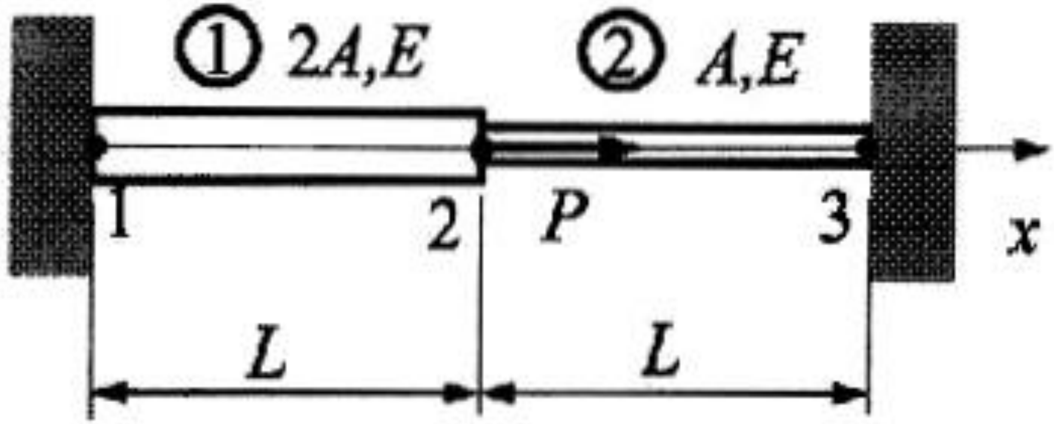
- To interpret the results use the FBD with your found values or use the computer program Algor

$$\sigma_1 = E\varepsilon_1 = E\mathbf{B}_1\mathbf{u}_1 = E\begin{bmatrix} -1/L & 1/L \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

FEM Steps (Displacement Method)

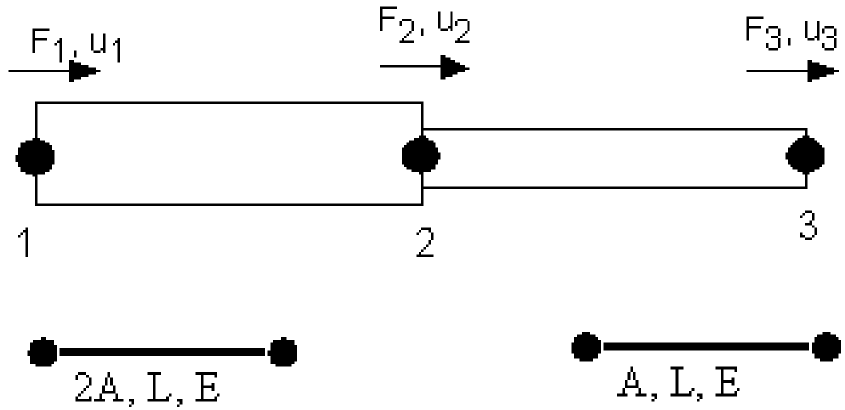
- Discretize into finite elements, Identify nodes & elements
- Develop element stiffness matrices $[k_e]$ for all elements
- Assemble element stiffness matrices to get the global stiffness matrix
- Apply kinematic boundary conditions
- Solve for displacements
- Finally solve for element forces and stresses by picking proper rows

Example



Problem: Find the stresses in the two bar assembly which is loaded with force P, and constrained at the two ends, as shown in the figure.

Solution: Use two 1-D bar elements.



$$U(x) = \alpha_1 + \alpha_2 \cdot x$$

$$U(x) = \alpha_2 + \alpha_3 \cdot x$$

Element 1,

$$\mathbf{k}_1 = \frac{2EA}{L}\begin{array}{c} \begin{array}{cc} u_1 & u_2 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \end{array}$$

Element 2,

$$\mathbf{k}_2 = \frac{EA}{L}\begin{array}{c} \begin{array}{cc} u_2 & u_3 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ 2 \\ 3 \end{array}$$

- We combine the two stiffness matrices into the global matrix.

$$[K] = \frac{EA}{L}\begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

Load and boundary conditions (BC) are,

$$u_1 = u_3 = 0, \qquad F_2 = P$$

FE equation becomes,

$$\frac{EA}{L}\begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ u_2 \\ 0 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ P \\ F_3 \end{Bmatrix}$$

Deleting the 1st row and column, and the 3rd row and column, we obtain,

$$\frac{EA}{L}[3]\{u_2\} = \{P\}$$

Thus,

$$u_2 = \frac{PL}{3EA}$$

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \frac{PL}{3EA}\begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}$$

- Now that the displacement at u_2 has been obtained, the end forces and stress values can be obtained by reverting back to the individual element stiffness matrices
- For the stress, you only need to look at the individual node of the stifness equation

Reactions

$$\{F_1\} = \frac{AE}{L}\begin{bmatrix} 2 & -2 & 0 \end{bmatrix} \frac{PL}{3AE}\begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} = -\frac{2P}{3}$$

$$\{F_3\} = \frac{AE}{L}\begin{bmatrix} 0 & -1 & 1 \end{bmatrix} \frac{PL}{3AE}\begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} = -\frac{P}{3}$$

Element Forces

Element 1

$$\begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} = \frac{2AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\frac{2AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\frac{PL}{3AE}\begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} -2p/3 \\ 2p/3 \end{Bmatrix}$$

Element 2

$$\begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

$$= \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\frac{PL}{3AE}\begin{Bmatrix} 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} p/3 \\ -p/3 \end{Bmatrix}$$

Element Stresses

Stress in element 1 is

$$\sigma_1 = E\varepsilon_1 = E\mathbf{B}_1\mathbf{u}_1 = E\left[-1/L \quad 1/L\right]\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$= E\frac{u_2 - u_1}{L} = \frac{E}{L}\left(\frac{PL}{3EA} - 0\right) = \frac{P}{3A}$$ (member is in tensior

Similarly, stress in element 2 is

$$\sigma_2 = E\varepsilon_2 = E\mathbf{B}_2\mathbf{u}_2 = E\left[-1/L \quad 1/L\right]\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

$$= E\frac{u_3 - u_2}{L} = \frac{E}{L}\left(0 - \frac{PL}{3EA}\right) = -\frac{P}{3A}$$

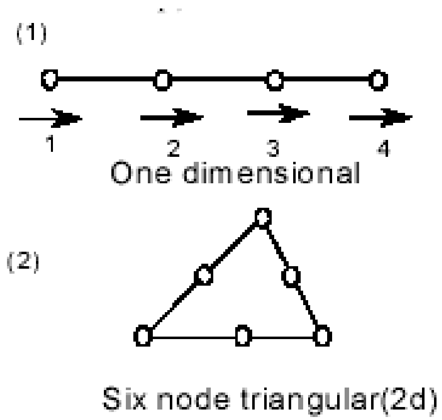
which indicates that bar 2 is in compression.

Final Notes

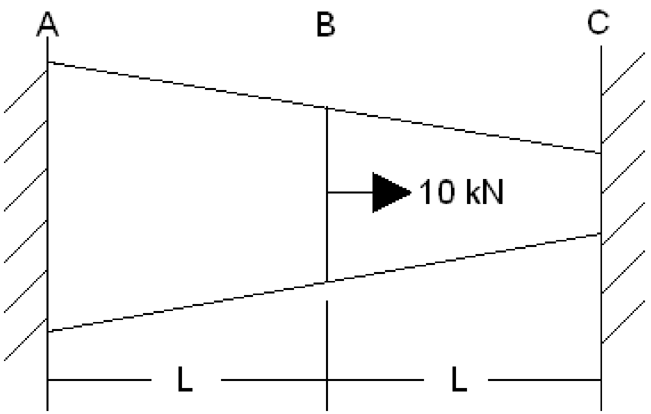
- For this case, the calculated stresses in elements 1 & 2 are exact within the linear theory for 1-D bar structures. Smaller finite elements will not help
- For tapered bars, averaged values of the cross-sectional areas should be used for the elements.
- The displacements must be found first to find the stresses, since we are using the displacement based FEM

Assignment

- Write the displacement functions for the following elements:



- Analyze the bar shown below for:
  - (a) Displacement at B
  - (b) End Forces
  - (c) Average Stresses in bar AB & BC



Section	Area (mm^2)
A	30
B	20
C	10

E = 200 GPa

Need for Computational Methods

- **Solutions Using Either Strength of Materials or Theory of Elasticity Are Normally Accomplished for Regions and Loadings With Relatively Simple Geometry**

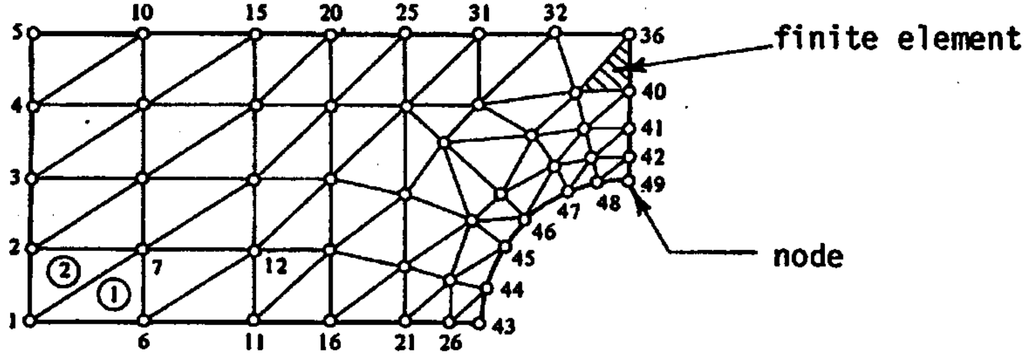
- **Many Applicaitons Involve Cases with Complex Shape, Boundary Conditions and Material Behavior**

- **Therefore a Gap Exists Between What Is Needed in Applications and What Can Be Solved by Analytical Closed-form Methods**

- **This Has Lead to the Development of Several Numerical/Computational Schemes Including: Finite Difference, Finite Element and Boundary Element Methods**

Introduction to Finite Element Analysis

The finite element method is a computational scheme to solve field problems in engineering and science. The technique has very wide application, and has been used on problems involving *stress analysis, fluid mechanics, heat transfer, diffusion, vibrations, electrical and magnetic fields*, etc. The fundamental concept involves dividing the body under study into a finite number of pieces (subdomains) called *elements* (see Figure). Particular assumptions are then made on the variation of the unknown dependent variable(s) across each element using so-called *interpolation or approximation functions*. This approximated variation is quantified in terms of solution values at special element locations called *nodes*. Through this discretization process, the method sets up an algebraic system of equations for unknown nodal values which approximate the continuous solution. Because element size, shape and approximating scheme can be varied to suit the problem, the method can accurately simulate solutions to problems of complex geometry and loading and thus this technique has become a very useful and practical tool.



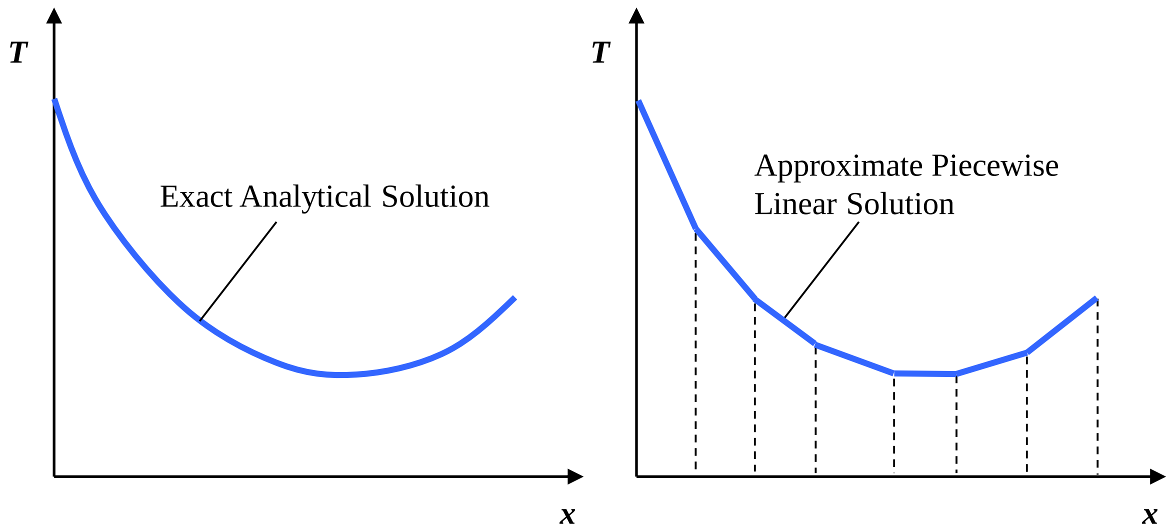
Advantages of Finite Element Analysis

- **Models Bodies of Complex Shape**
- **Can Handle General Loading/Boundary Conditions**
- **Models Bodies Composed of Composite and Multiphase Materials**
- **Model is Easily Refined for Improved Accuracy by Varying Element Size and Type (Approximation Scheme)**
- **Time Dependent and Dynamic Effects Can Be Included**
- **Can Handle a Variety Nonlinear Effects Including Material Behavior, Large Deformations, Boundary Conditions, Etc.**

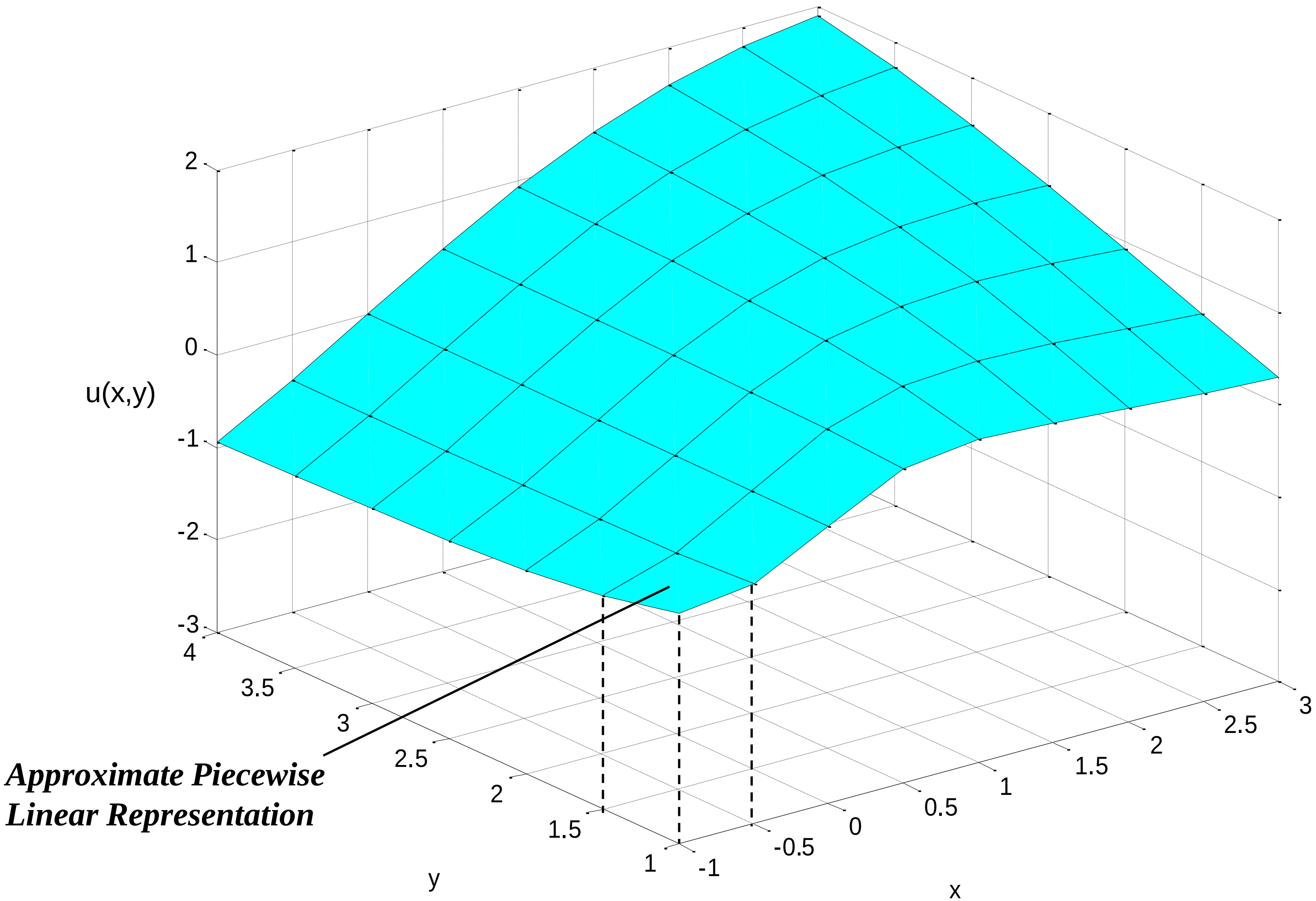
Basic Concept of the Finite Element Method

Any continuous solution field such as stress, displacement, temperature, pressure, etc. can be approximated by a discrete model composed of a set of piecewise continuous functions defined over a finite number of subdomains.

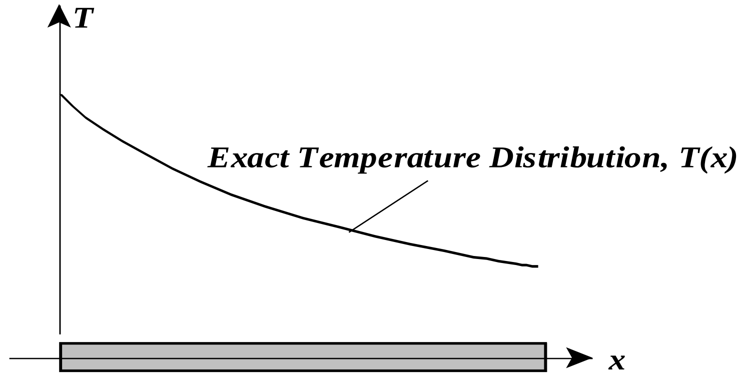
<u>One-Dimensional Temperature Distribution</u>



Two-Dimensional Discretization

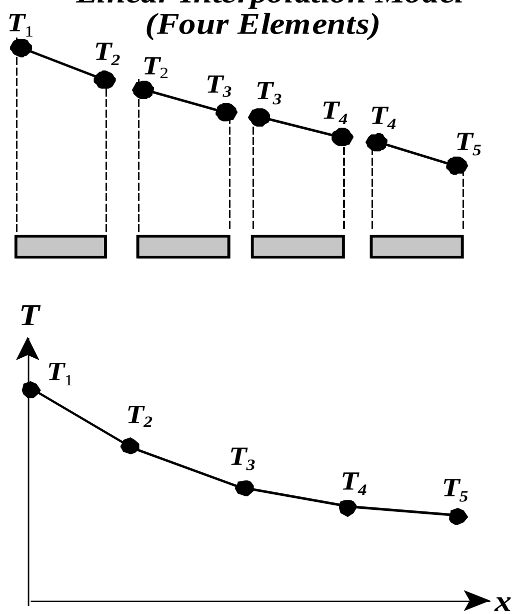


Discretization Concepts



Finite Element Discretization

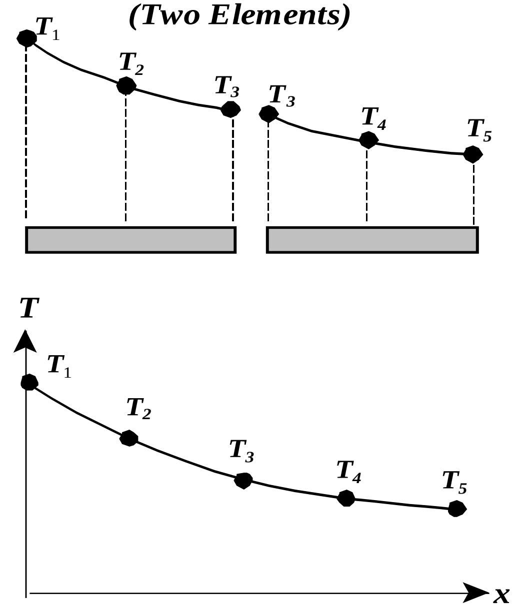
Linear Interpolation Model
(Four Elements)



Piecewise Linear Approximation

Temperature Continuous but with Discontinuous Temperature Gradients

Quadratic Interpolation Model
(Two Elements)



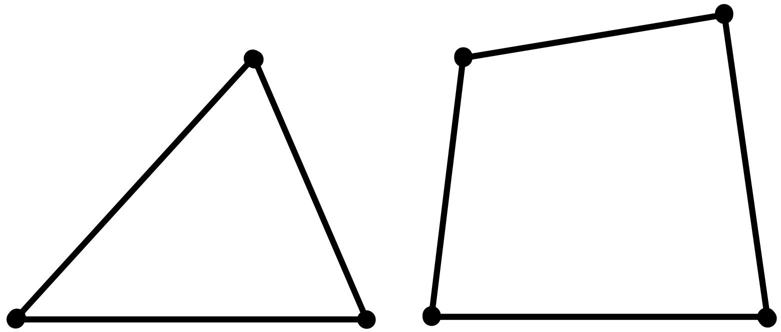
Piecewise Quadratic Approximation

Temperature and Temperature Gradients Continuous

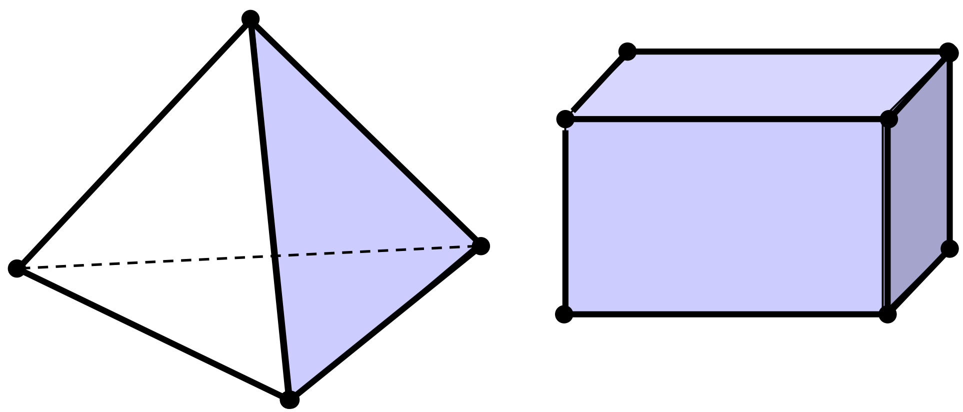
Common Types of Elements

<u>One-Dimensional Elements</u>
Line
Rods, Beams, Trusses, Frames

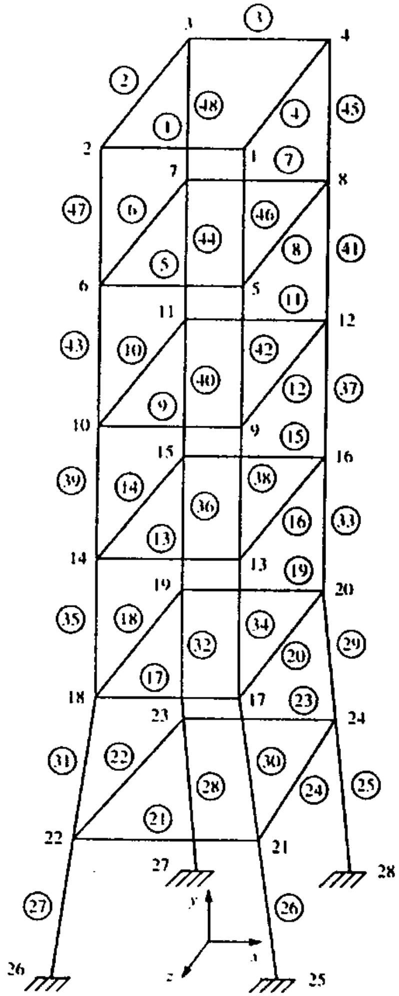
<u>Two-Dimensional Elements</u>
Triangular, Quadrilateral
Plates, Shells, 2-D Continua



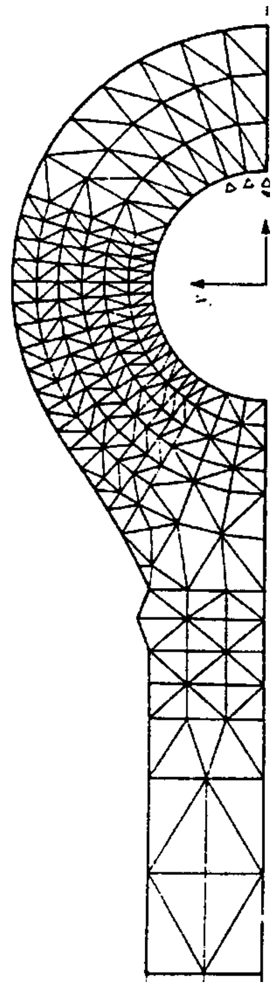
<u>Three-Dimensional Elements</u>
Tetrahedral, Rectangular Prism (Brick)
3-D Continua



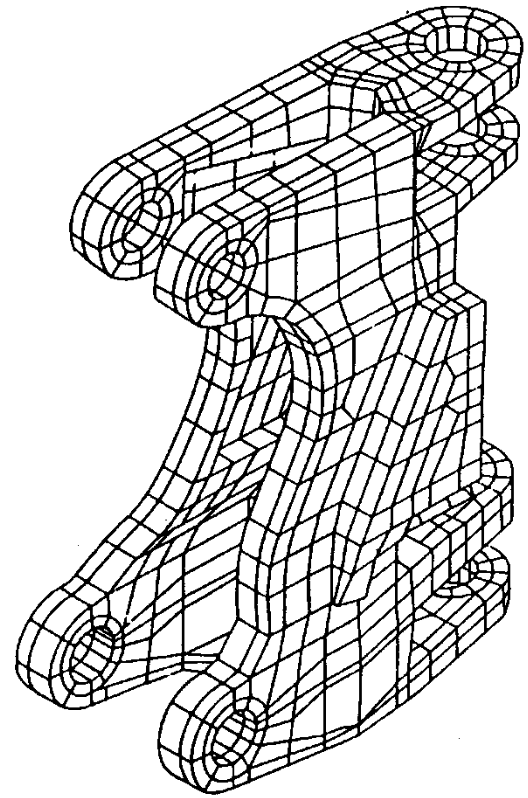
Discretization Examples



One-Dimensional Frame Elements



Two-Dimensional Triangular Elements



Three-Dimensional Brick Elements

Basic Steps in the Finite Element Method
Time Independent Problems

- **Domain Discretization**
- **Select Element Type (Shape and Approximation)**
- **Derive Element Equations (Variational and Energy Methods)**
- **Assemble Element Equations to Form Global System**

$$[K]\{U\} = \{F\}$$

[K] = Stiffness or Property Matrix
{U} = Nodal Displacement Vector
{F} = Nodal Force Vector

- **Incorporate Boundary and Initial Conditions**
- **Solve Assembled System of Equations for Unknown Nodal Displacements and Secondary Unknowns of Stress and Strain Values**

Common Sources of Error in FEA

- **Domain Approximation**
- **Element Interpolation/Approximation**
- **Numerical Integration Errors (Including Spatial and Time Integration)**
- **Computer Errors (Round-Off, Etc.,)**

Measures of Accuracy in FEA

<u>Accuracy</u>

Error = |(Exact Solution)-(FEM Solution)|

<u>Convergence</u>

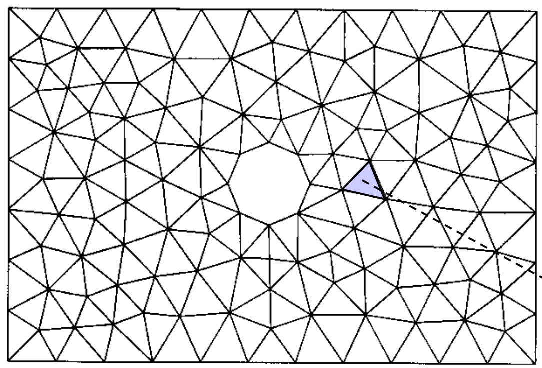
Limit of Error as:

Number of Elements (*h-convergence*)
or
Approximation Order (*p-convergence*)

Increases

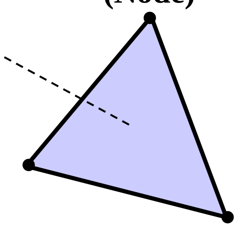
Ideally, Error $\rightarrow$ 0 as Number of Elements or Approximation Order $\rightarrow \infty$

Two-Dimensional Discretization Refinement

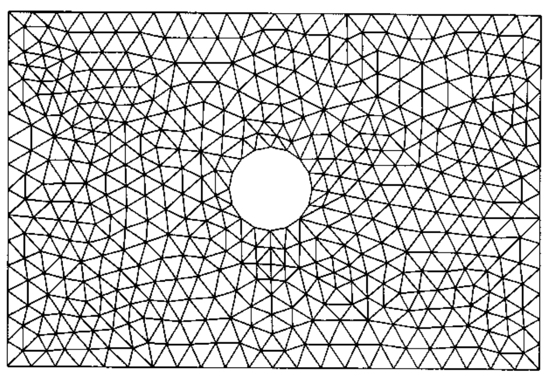


(Discretization with 228 Elements)

(Node)



(Triangular Element)

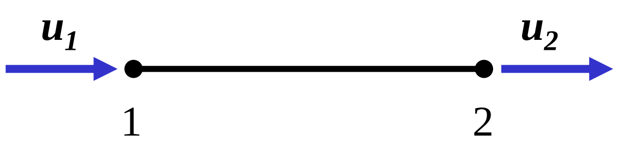


(Discretization with 912 Elements)

One Dimensional Examples
Static Case

Bar Element

Uniaxial Deformation of Bars
Using Strength of Materials Theory



Differential Equation :

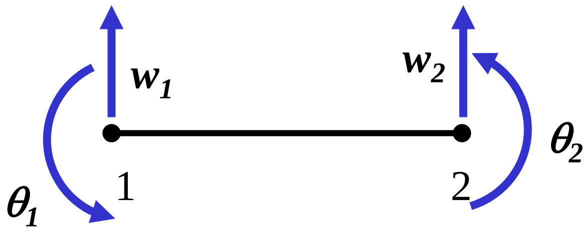
$$-\frac{d}{dx}(au) + cu - q = 0$$

Boundary Condtions Specification :

$$u \ , \ a\frac{du}{dx}$$

Beam Element

Deflection of Elastic Beams
Using Euler-Bernouli Theory



Differential Equation :

$$-\frac{d^2}{dx^2}(b\frac{d^2w}{dx^2}) = f(x)$$

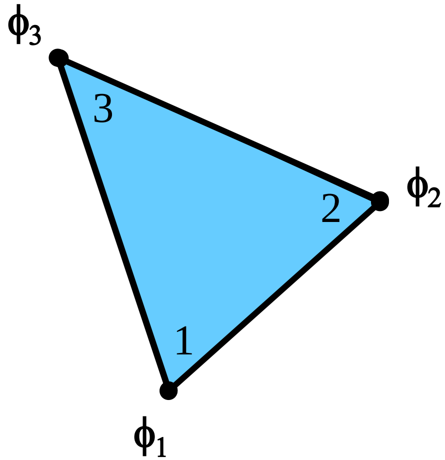
Boundary Condtions Specification :

$$w \ , \ \frac{dw}{dx} \ , \ b\frac{d^2w}{dx^2} \ , \ \frac{d}{dx}(b\frac{d^2w}{dx^2})$$

Two Dimensional Examples

Triangular Element

Scalar-Valued, Two-Dimensional Field Problems



Example Differential Equation :

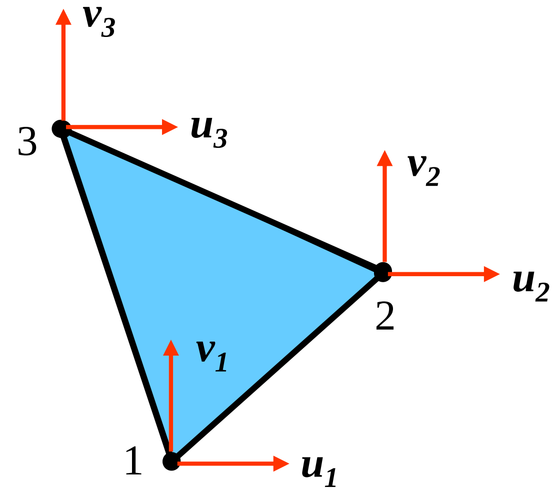
$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=f(x,y)$$

Boundary Condtions Specification :

$$\phi\ ,\ \frac{d\phi}{dn}=\frac{\partial\phi}{\partial x}n_x+\frac{\partial\phi}{\partial y}n_y$$

Triangular Element

Vector/Tensor-Valued, Two-Dimensional Field Problems



Elasticity Field Equations in Terms of Displacements

$$\mu\nabla^2 u+\frac{E}{2(1-\nu)}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+F_x=0$$

$$\mu\nabla^2 v+\frac{E}{2(1-\nu)}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+F_y=0$$

Boundary Conditons

$$T_x=\left(C_{11}\frac{\partial u}{\partial x}+C_{12}\frac{\partial v}{\partial y}\right)n_x+C_{66}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)n_y$$

$$T_y=C_{66}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)n_x+\left(C_{12}\frac{\partial u}{\partial x}+C_{22}\frac{\partial v}{\partial y}\right)n_y$$

Development of Finite Element Equation

- **The Finite Element Equation Must Incorporate the Appropriate Physics of the Problem**
- **For Problems in Structural Solid Mechanics, the Appropriate Physics Comes from Either Strength of Materials or Theory of Elasticity**
- **FEM Equations are Commonly Developed Using *Direct*, *Variational-Virtual Work* or *Weighted Residual* Methods**

Direct Method

Based on physical reasoning and limited to simple cases, this method is worth studying because it enhances physical understanding of the process

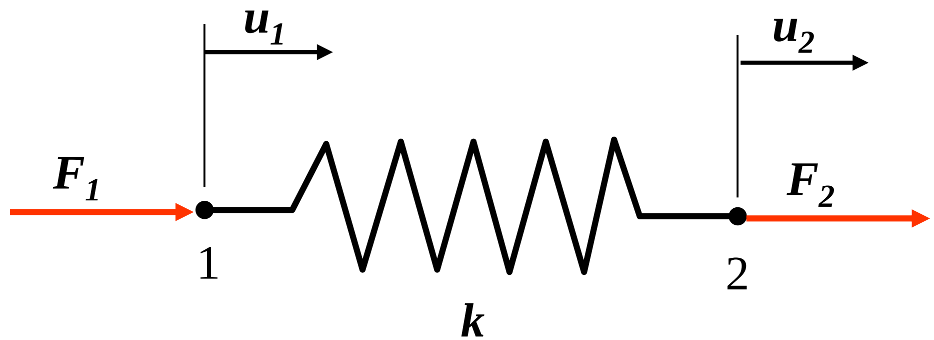
Variational-Virtual Work Method

Based on the concept of virtual displacements, leads to relations between internal and external virtual work and to minimization of system potential energy for equilibrium

Weighted Residual Method

Starting with the governing differential equation, special mathematical operations develop the "weak form" that can be incorporated into a FEM equation. This method is particularly suited for problems that have no variational statement. For stress analysis problems, a Ritz-Galerkin WRM will yield a result identical to that found by variational methods.

Simple Element Equation Example Direct Stiffness Derivation



$$\text{Equilibrium at Node 1} \Rightarrow F_1 = ku_1 - ku_2$$

$$\text{Equilibrium at Node 2} \Rightarrow F_2 = -ku_1 + ku_2$$

or in Matrix Form

$$\begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

Stiffness Matrix

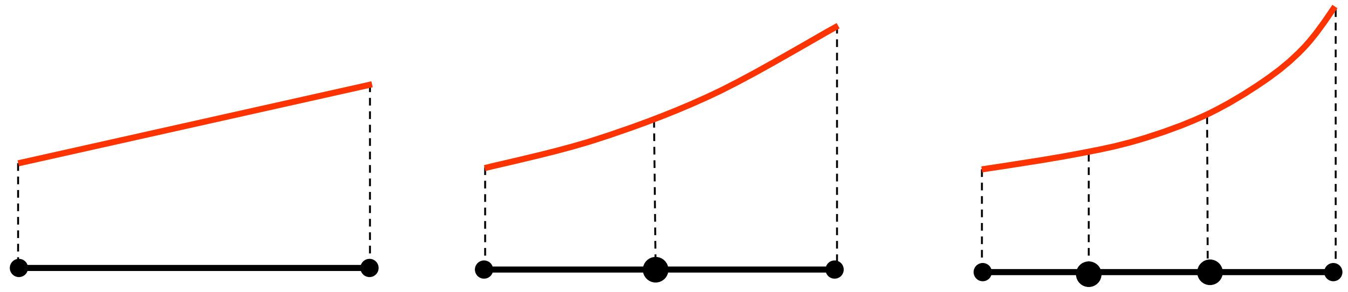
Nodal Force Vector

$$[K]\{u\} = \{F\}$$

Common Approximation Schemes One-Dimensional Examples

Polynomial Approximation

Most often polynomials are used to construct approximation functions for each element. Depending on the order of approximation, different numbers of element parameters are needed to construct the appropriate function.



Linear **Quadratic** **Cubic**

Special Approximation

For some cases (e.g. infinite elements, crack or other singular elements) the approximation function is chosen to have special properties as determined from theoretical considerations

One-Dimensional Bar Element

$$\text{Approximation}: u = \sum_k \psi_k(x) u_k = [\boldsymbol{N}]\{\boldsymbol{d}\}$$

$$\text{Strain}: \; e = \frac{du}{dx} = \sum_k \frac{d}{dx}\psi_k(x) u_k = \frac{d[\boldsymbol{N}]}{dx}\{\boldsymbol{d}\} = [\boldsymbol{B}]\{\boldsymbol{d}\}$$

$$\text{Stress-Strain Law}: \; \sigma = Ee = E[\boldsymbol{B}]\{\boldsymbol{d}\}$$

$$\int_\Omega \sigma \delta e dV = P_i u_i + P_j u_j + \int_\Omega f \delta u dV \;\; \Rightarrow$$

$$\{\boldsymbol{\delta d}\}^T \int_0^L A[\boldsymbol{B}]^T E[\boldsymbol{B}] dx \{\boldsymbol{d}\} = \{\boldsymbol{\delta d}\}^T \begin{Bmatrix} P_i \\ P_j \end{Bmatrix} + \{\boldsymbol{\delta d}\}^T \int_0^L A[\boldsymbol{N}]^T f dx \;\; \Rightarrow$$

$$\int_0^L A[\boldsymbol{B}]^T E[\boldsymbol{B}] dx \{\boldsymbol{d}\} = \{\boldsymbol{P}\} + \int_0^L A[\boldsymbol{N}]^T f dx$$

$$\boxed{[\boldsymbol{K}]\{\boldsymbol{d}\} = \{\boldsymbol{F}\}}$$

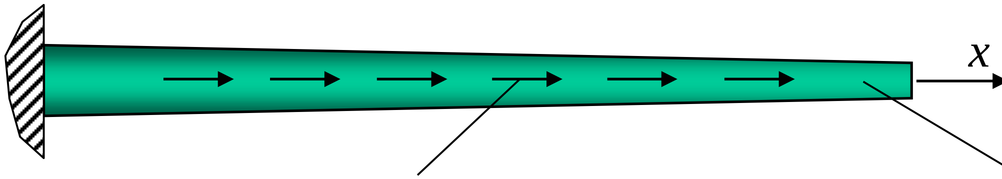
$$[K] = \int_0^L A[\boldsymbol{B}]^T E[\boldsymbol{B}] dx = \text{Stiffness Matrix}$$

$$\{\boldsymbol{F}\} = \begin{Bmatrix} P_i \\ P_j \end{Bmatrix} + \int_0^L A[\boldsymbol{N}]^T f dx = \text{Loading Vector}$$

$$\{\boldsymbol{d}\} = \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} = \text{Nodal Displacement Vector}$$

One-Dimensional Bar Element

Axial Deformation of an Elastic Bar

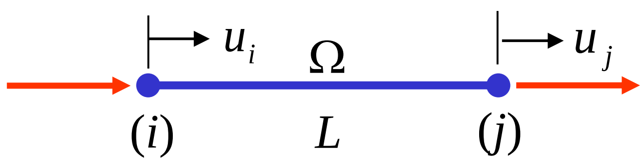


f(x) = Distributed Loading

A = Cross-sectional Area
E = Elastic Modulus

Typical Bar Element

$$P_i = -AE\frac{du_i}{dx}$$



$$P_j = -AE\frac{du_j}{dx}$$

(Two Degrees of Freedom)

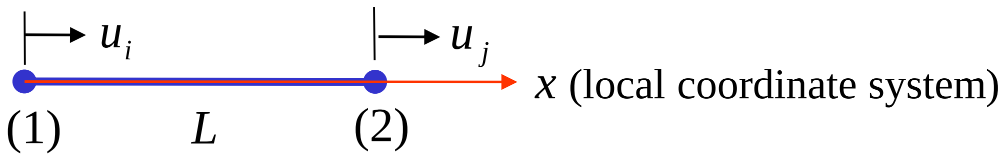
Virtual Strain Energy = Virtual Work Done by Surface and Body Forces

$$\int_V \sigma_{ij}\delta e_{ij}dV = \int_{S_t} T_i^n \delta u_i dS + \int_V F_i \delta u_i dV$$

For One-Dimensional Case

$$\int_\Omega \sigma\delta e dV = P_i u_i + P_j u_j + \int_\Omega f\delta u dV$$

Linear Approximation Scheme



Approximate Elastic Displacement

$$u = a_1 + a_2 x \quad \Rightarrow \quad \begin{aligned} u_1 &= a_1 \\ u_2 &= a_1 + a_2 L \end{aligned}$$

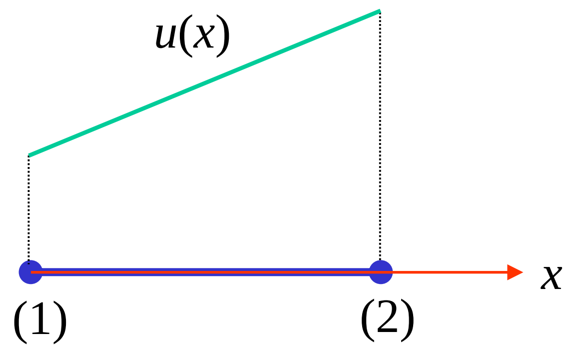
$$\Rightarrow u = u_1 + \frac{u_2 - u_1}{L} x = \left(1 - \frac{x}{L}\right) u_1 + \left(\frac{x}{L}\right) u_2$$

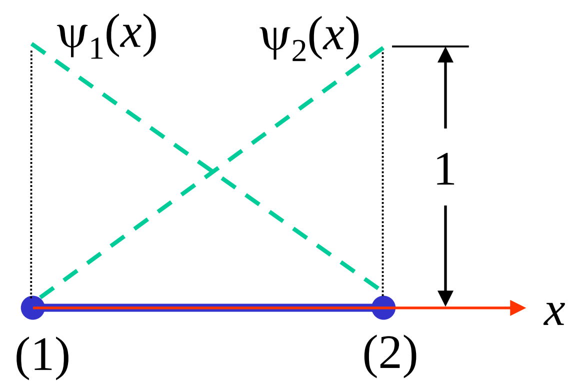
$$= \psi_1(x) u_1 + \psi_2(x) u_2$$

$$u = \begin{bmatrix} \psi_1 & \psi_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} 1 - \frac{x}{L} & \frac{x}{L} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = [\boldsymbol{N}]\{\boldsymbol{d}\}$$

$[\boldsymbol{N}]$ = Approximation Function Matrix

$\{\boldsymbol{d}\}$ = Nodal Displacement Vector





$\psi_k(x)$ – Lagrange Interpolation Functions

Element Equation

Linear Approximation Scheme, Constant Properties

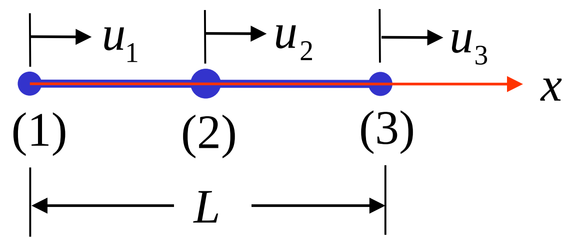
$$[K]=\int_0^L A[\boldsymbol{B}]^T E[\boldsymbol{B}]dx = AE[\boldsymbol{B}]^T[\boldsymbol{B}]\int_0^L dx = AE\begin{Bmatrix}-\dfrac{1}{L}\\ \dfrac{1}{L}\end{Bmatrix}\begin{Bmatrix}-\dfrac{1}{L} & \dfrac{1}{L}\end{Bmatrix}L=\frac{AE}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

$$\{\boldsymbol{F}\}=\begin{Bmatrix}P_1\\ P_2\end{Bmatrix}+\int_0^L A[\boldsymbol{N}]^T\, f dx=\begin{Bmatrix}P_1\\ P_2\end{Bmatrix}+Af_o\int_0^L\begin{Bmatrix}-\dfrac{x}{L}\\ \dfrac{x}{L}\end{Bmatrix}dx=\begin{Bmatrix}P_1\\ P_2\end{Bmatrix}+\frac{Af_oL}{2}\begin{Bmatrix}1\\ 1\end{Bmatrix}$$

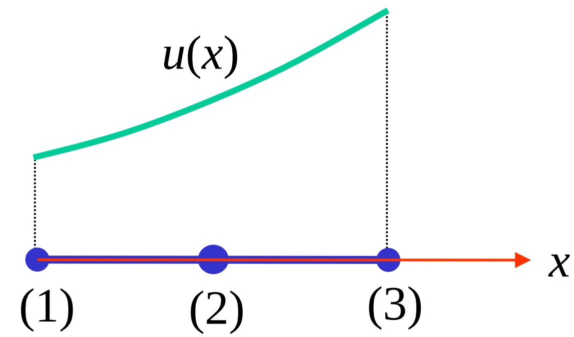
$$\{\boldsymbol{d}\}=\begin{Bmatrix}u_1\\ u_2\end{Bmatrix}=\text{Nodal Displacement Vector}$$

$$[\boldsymbol{K}]\{\boldsymbol{d}\}=\{\boldsymbol{F}\}\Rightarrow\ \frac{AE}{L}\begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix}=\begin{Bmatrix}P_1\\ P_2\end{Bmatrix}+\frac{Af_oL}{2}\begin{Bmatrix}1\\ 1\end{Bmatrix}$$

Quadratic Approximation Scheme



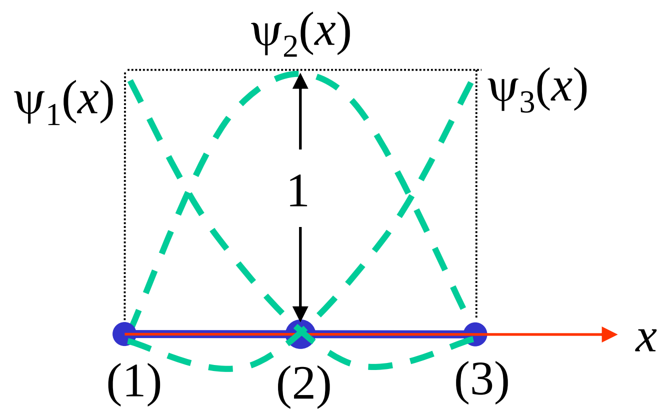
Approximate Elastic Displacement



$$u = a_1 + a_2 x + a_3 x^2 \quad \Rightarrow \quad \begin{aligned} u_1 &= a_1 \\ u_2 &= a_1 + a_2 \frac{L}{2} + a_3 \frac{L^2}{4} \\ u_3 &= a_1 + a_2 L + a_3 L^2 \end{aligned}$$

$$u = \psi_1(x) u_1 + \psi_2(x) u_2 + \psi_3(x) u_3$$

$$u = \begin{bmatrix} \psi_1 & \psi_2 & \psi_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = [\boldsymbol{N}]\{\boldsymbol{d}\}$$



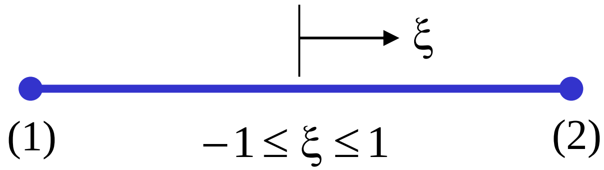
Element Equation

$$\frac{AE}{3L} \begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

Lagrange Interpolation Functions

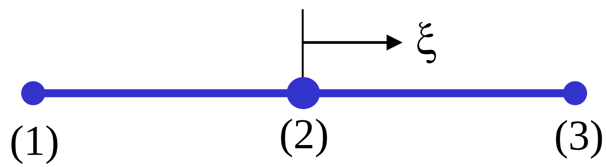
Using Natural or Normalized Coordinates

$$\psi_i(\xi_j) = \begin{cases} 1 \, , \; i = j \\ 0 \, , \; i \neq j \end{cases}$$



$$\psi_1 = \frac{1}{2}(1-\xi)$$

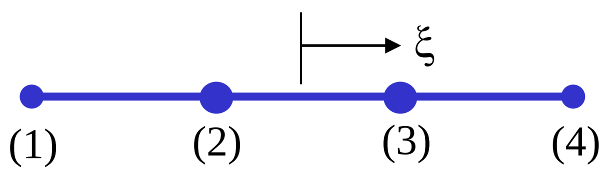
$$\psi_2 = \frac{1}{2}(1+\xi)$$



$$\psi_1 = -\frac{1}{2}\xi(1-\xi)$$

$$\psi_2 = (1-\xi)(1+\xi)$$

$$\psi_3 = \frac{1}{2}\xi(1+\xi)$$



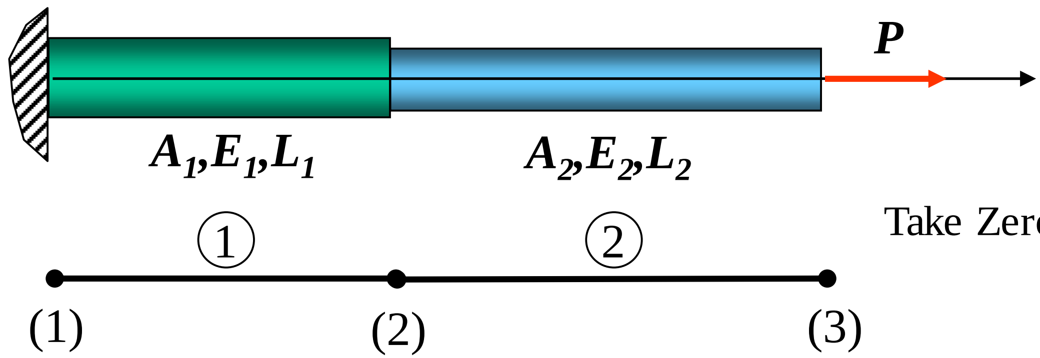
$$\psi_1 = -\frac{9}{16}(1-\xi)(\frac{1}{3}+\xi)(\frac{1}{3}-\xi)$$

$$\psi_2 = \frac{27}{16}(1-\xi)(1+\xi)(\frac{1}{3}-\xi)$$

$$\psi_3 = \frac{27}{16}(1-\xi)(1+\xi)(\frac{1}{3}+\xi)$$

$$\psi_4 = -\frac{9}{16}(\frac{1}{3}+\xi)(\frac{1}{3}-\xi)(1+\xi)$$

Simple Example



Take Zero Distributed Loading

$$f = 0$$

Global Equation Element 1

$$\frac{A_1E_1}{L_1}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} P_1^{(1)} \\ P_2^{(1)} \\ 0 \end{Bmatrix}$$

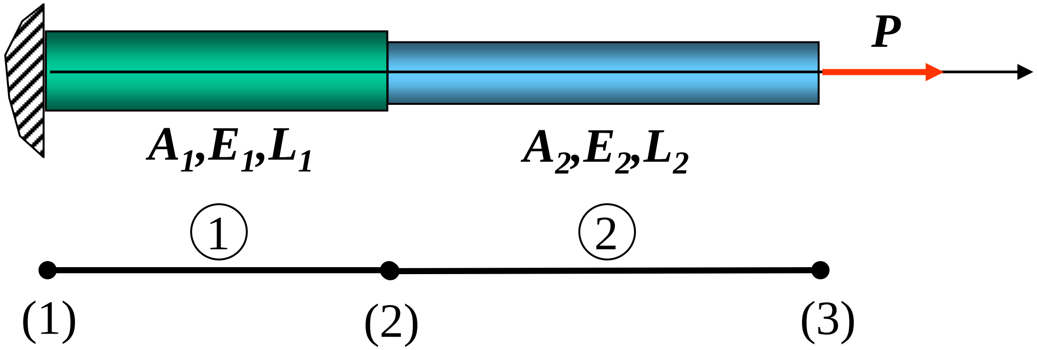
Global Equation Element 2

$$\frac{A_2E_2}{L_2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ P_1^{(2)} \\ P_2^{(2)} \end{Bmatrix}$$

Assembled Global System Equation

$$\begin{bmatrix} \frac{A_1E_1}{L_1} & -\frac{A_1E_1}{L_1} & 0 \\ -\frac{A_1E_1}{L_1} & \frac{A_1E_1}{L_1} + \frac{A_2E_2}{L_2} & -\frac{A_2E_2}{L_2} \\ 0 & -\frac{A_2E_2}{L_2} & \frac{A_2E_2}{L_2} \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} P_1^{(1)} \\ P_2^{(1)} + P_1^{(2)} \\ P_2^{(2)} \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix}$$

Simple Example Continued



Reduced Global System Equation

Boundary Conditions

$$U_1 = 0$$

$$P_2^{(2)} = P$$

$$P_2^{(1)} + P_1^{(2)} = 0$$

$$\begin{bmatrix} \frac{A_1E_1}{L_1} & -\frac{A_1E_1}{L_1} & 0 \\ -\frac{A_1E_1}{L_1} & \frac{A_1E_1}{L_1} + \frac{A_2E_2}{L_2} & -\frac{A_2E_2}{L_2} \\ 0 & -\frac{A_2E_2}{L_2} & \frac{A_2E_2}{L_2} \end{bmatrix} \begin{Bmatrix} 0 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} P_1^{(1)} \\ 0 \\ P \end{Bmatrix}$$

$$\begin{bmatrix} \frac{A_1E_1}{L_1} + \frac{A_2E_2}{L_2} & -\frac{A_2E_2}{L_2} \\ -\frac{A_2E_2}{L_2} & \frac{A_2E_2}{L_2} \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ P \end{Bmatrix}$$

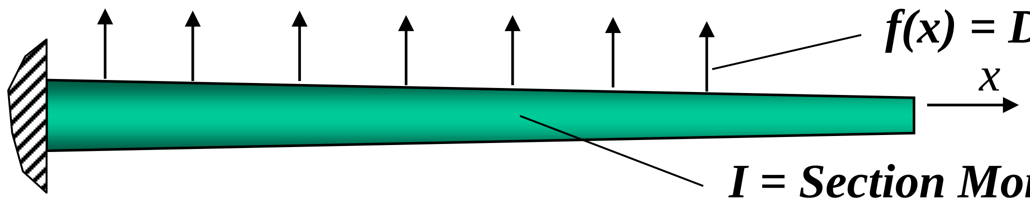
For Unifom Properties A, E, L

$$\frac{AE}{L}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ P \end{Bmatrix}$$

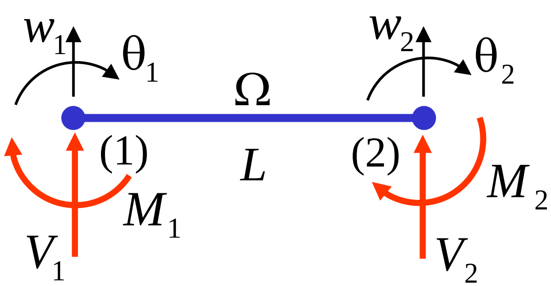
Solving $\Rightarrow U_2 = \frac{PL}{AE}\ ,\ U_3 = \frac{2PL}{AE}\ ,\ P_1^{(1)} = -P$

One-Dimensional Beam Element

Deflection of an Elastic Beam



Typical Beam Element



(Four Degrees of Freedom)

$$Q_1 = \frac{d}{dx}\left(EI\,\frac{d^2w}{dx^2}\right)_1 \ , \ Q_2 = \left(EI\,\frac{d^2w}{dx^2}\right)_1$$

$$Q_3 = -\frac{d}{dx}\left(EI\,\frac{d^2w}{dx^2}\right)_2 \ , \ Q_4 = -\left(EI\,\frac{d^2w}{dx^2}\right)_2$$

$$u_1 = w_1 \ , \ u_2 = \theta_1 = -\left.\frac{dw}{dx}\right|_1 \ , \ u_3 = w_2 \ , \ u_4 = \theta_2 = -\left.\frac{dw}{dx}\right|_2$$

Virtual Strain Energy = Virtual Work Done by Surface and Body Forces

$$\int_\Omega \sigma\delta e dV = Q_1u_1 + Q_2u_2 + Q_3u_3 + Q_4w_4 + \int_\Omega f\delta w dV \Rightarrow$$

$$EI\int_0^L [\boldsymbol{B}]^T[\boldsymbol{B}]dx\{\boldsymbol{d}\} = Q_1u_1 + Q_2u_2 + Q_3u_3 + Q_4w_4 + \int_0^L f[\boldsymbol{N}]^T dV$$

Beam Approximation Functions

To approximate deflection and slope at each node requires approximation of the form

$$w(x) = c_1 + c_2 x + c_3 x^2 + c_4 x^3$$

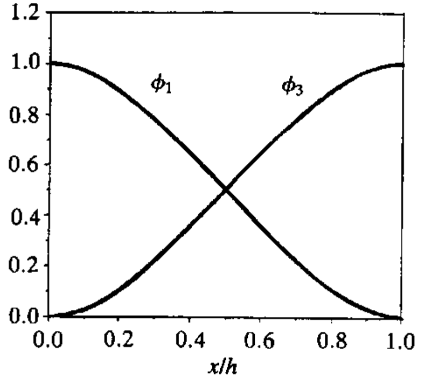
Evaluating deflection and slope at each node allows the determination of c_i thus leading to

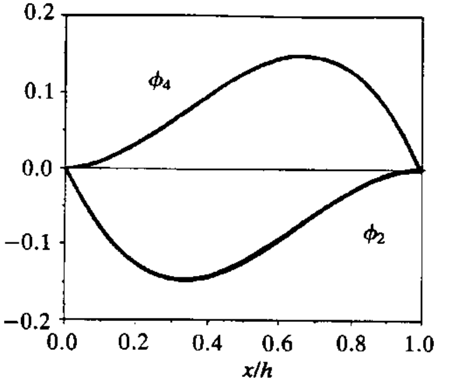
$$w(x) = \phi_1(x)u_1 + \phi_2(x)u_2 + \phi_3(x)u_3 + \phi_4(x)u_4 \ ,$$

where ϕ_i are the Hermite Cubic Approximation Functions

$$\phi_1^e = 1 - 3\left(\frac{\bar{x}}{h_e}\right)^2 + 2\left(\frac{\bar{x}}{h_e}\right)^3, \qquad \phi_2^e = -\bar{x}\left(1 - \frac{\bar{x}}{h_e}\right)^2$$

$$\phi_3^e = 3\left(\frac{\bar{x}}{h_e}\right)^2 - 2\left(\frac{\bar{x}}{h_e}\right)^3, \qquad \phi_4^e = -\bar{x}\left[\left(\frac{\bar{x}}{h_e}\right)^2 - \frac{\bar{x}}{h_e}\right]$$





Beam Element Equation

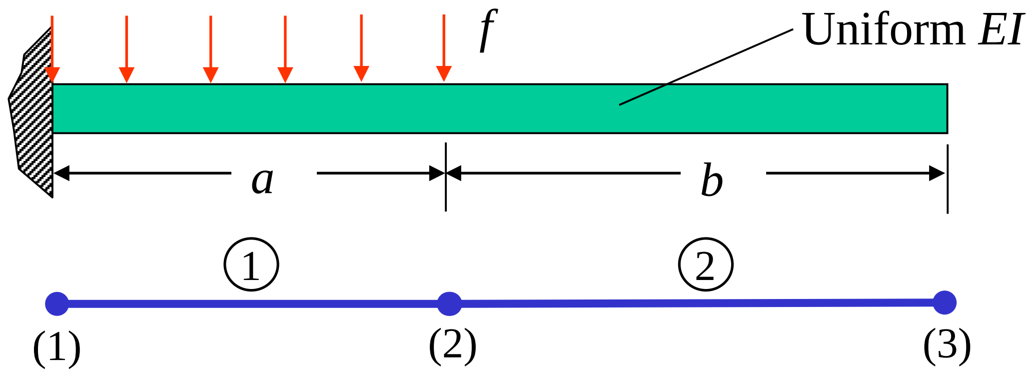
$$EI\int_0^L [\boldsymbol{B}]^T [\boldsymbol{B}] dx \{\boldsymbol{d}\} = Q_1u_1 + Q_2u_2 + Q_3u_3 + Q_4w_4 + \int_0^L f[\boldsymbol{N}]^T dV$$

$$\{\boldsymbol{d}\} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} \qquad [\boldsymbol{B}] = \frac{d[\boldsymbol{N}]}{dx} = [\frac{d\phi_1}{dx}\ \frac{d\phi_2}{dx}\ \frac{d\phi_3}{dx}\ \frac{d\phi_4}{dx}]$$

$$[\boldsymbol{K}] = EI\int_0^L [\boldsymbol{B}]^T [\boldsymbol{B}] dx = \frac{2EI}{L^3}\begin{bmatrix} 6 & -3L & -6 & -3L \\ -3L & 2L^2 & 3L & L^2 \\ -6 & 3L & 6 & 3L \\ -3L & L^2 & 3L & 2L^2 \end{bmatrix} \qquad \int_0^L f[\boldsymbol{N}]^T dx = f\int_0^L \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{Bmatrix} dx = \frac{fL}{12}\begin{Bmatrix} 6 \\ -L \\ 6 \\ L \end{Bmatrix}$$

$$\frac{2EI}{L^3}\begin{bmatrix} 6 & -3L & -6 & -3L \\ -3L & 2L^2 & 3L & L^2 \\ -6 & 3L & 6 & 3L \\ -3L & L^2 & 3L & 2L^2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix} + \frac{fL}{12}\begin{Bmatrix} 6 \\ -L \\ 6 \\ L \end{Bmatrix}$$

FEA Beam Problem



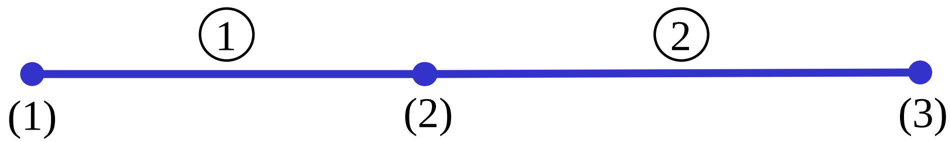
Element 1

$$2EI\begin{bmatrix} 6/a^3 & -3/a^2 & -6/a^3 & -3/a^2 & 0 & 0 \\ -3/a^2 & 2/a & 3/a^2 & 1/a & 0 & 0 \\ -6/a^3 & 3/a^2 & 6/a^3 & 3/a^2 & 0 & 0 \\ -3/a^2 & 1/a & 3/a^2 & 2/a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix} = -\frac{fa}{12}\begin{Bmatrix} 6 \\ -a \\ 6 \\ a \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} Q_1^{(1)} \\ Q_2^{(1)} \\ Q_3^{(1)} \\ Q_4^{(1)} \\ 0 \\ 0 \end{Bmatrix}$$

Element 2

$$2EI\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6/b^3 & -3/b^2 & -6/b^3 & -3/b^2 \\ 0 & 0 & -3/b^2 & 2/b & 3/b^2 & 1/b \\ 0 & 0 & -6/b^3 & 3/b^2 & 6/b^3 & 3/b^2 \\ 0 & 0 & -3/b^2 & 1/b & 3/b^2 & 2/b \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ Q_1^{(2)} \\ Q_2^{(2)} \\ Q_3^{(2)} \\ Q_4^{(2)} \end{Bmatrix}$$

FEA Beam Problem



Global Assembled System

$$2EI\begin{bmatrix} 6/a^3 & -3/a^2 & -6/a^3 & -3/a^2 & 0 & 0 \\ \cdot & 2/a & 3/a^2 & 1/a & 0 & 0 \\ \cdot & \cdot & 6/a^3+6/b^3 & 3/a^2-3/b^2 & -6/a^3 & -3/a^2 \\ \cdot & \cdot & \cdot & 2/a+2/b & 3/a^2 & 1/a \\ \cdot & \cdot & \cdot & \cdot & 6/a^3 & 3/a^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 2/a \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix} = -\frac{fa}{12}\begin{Bmatrix} 6 \\ -a \\ 6 \\ a \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} Q_1^{(1)} \\ Q_2^{(1)} \\ Q_3^{(1)}+Q_1^{(2)} \\ Q_4^{(1)}+Q_2^{(2)} \\ Q_3^{(2)} \\ Q_4^{(2)} \end{Bmatrix}$$

Boundary Conditions

$$U_1 = w_1^{(1)} = 0 \;,\; U_2 = \theta_1^{(1)} = 0 \;,\; Q_3^{(2)} = Q_4^{(2)} = 0$$

Matching Conditions

$$Q_3^{(1)} + Q_1^{(2)} = 0 \;,\; Q_4^{(1)} + Q_2^{(2)} = 0$$

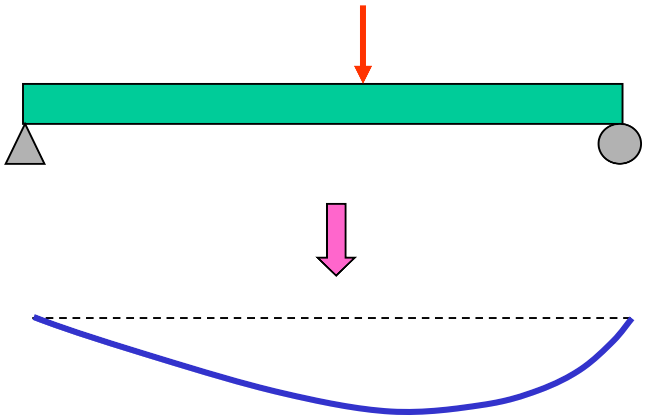
Reduced System

$$2EI\begin{bmatrix} 6/a^3+6/b^3 & 3/a^2-3/b^2 & -6/a^3 & -3/a^3 \\ \cdot & 2/a+2/b & 3/a^2 & 1/a \\ \cdot & \cdot & 6/a^3 & 3/a^2 \\ \cdot & \cdot & \cdot & 2/a \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = -\frac{fa}{12}\begin{Bmatrix} 6 \\ a \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

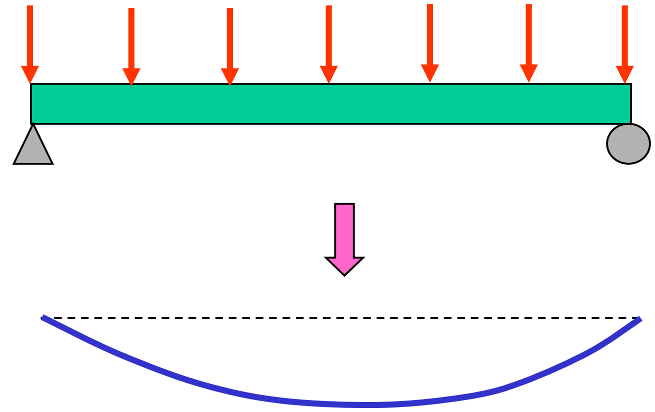
Solve System for Primary Unknowns U_1, U_2, U_3, U_4

Nodal Forces Q_1 and Q_2 Can Then Be Determined

Special Features of Beam FEA



Analytical Solution Gives
Cubic Deflection Curve

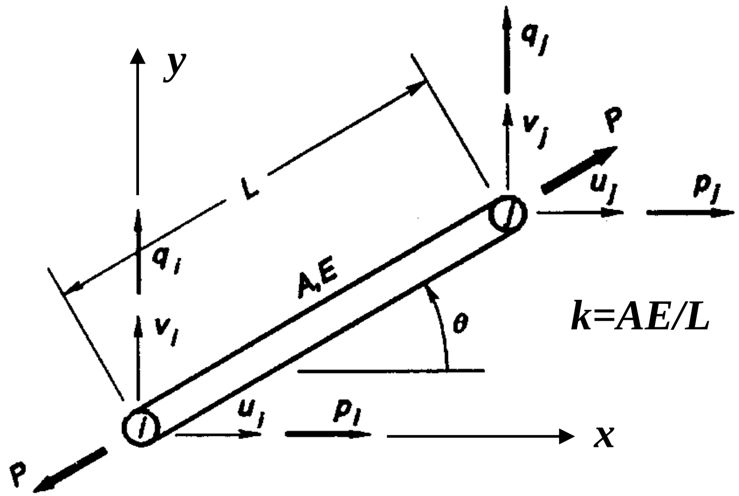


Analytical Solution Gives
Quartic Deflection Curve

FEA Using Hermit Cubic Interpolation
Will Yield Results That Match Exactly
With Cubic Analytical Solutions

Truss Element

Generalization of Bar Element With Arbitrary Orientation



k=AE/L

Basic Element Equation (θ = 0 case)

$$\begin{bmatrix} k & 0 & -k & 0 \\ 0 & 0 & 0 & 0 \\ -k & 0 & k & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \end{Bmatrix} = \begin{Bmatrix} -p_i \\ -q_i \\ -p_j \\ -q_j \end{Bmatrix}$$

Transformation for General Orientation

$$[T] = \begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix}$$

$$\{d\} = [T]\{d'\} \qquad \{f\} = [T]\{f'\}$$

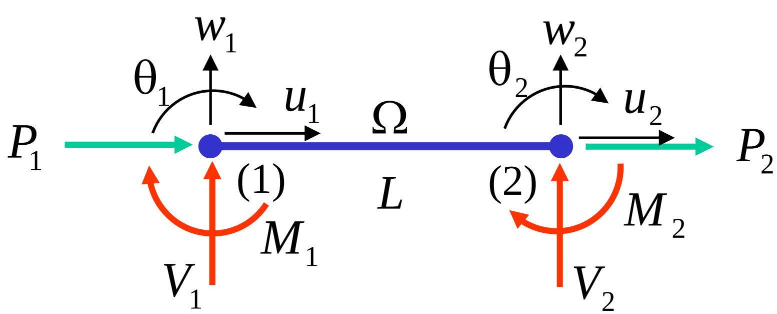
$$[k]\{d\} = \{f\} \Rightarrow [T]^{\mathrm{T}}[k][T]\{d'\} = \{f'\}$$

$s = \sin\theta$, $c = \cos\theta$

$$[k'] = [T]^{\mathrm{T}}[k][T] = k \begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}$$

Frame Element

Generalization of Bar and Beam Element with Arbitrary Orientation



$$\begin{bmatrix} \frac{AE}{L} & 0 & 0 & -\frac{AE}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \begin{Bmatrix} u_1 \\ w_1 \\ \theta_1 \\ u_2 \\ w_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ Q_1 \\ Q_2 \\ P_2 \\ Q_3 \\ Q_4 \end{Bmatrix}$$

Element Equation Can Then Be Rotated to Accommodate Arbitrary Orientation

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